Exam — Analysis (WBMA012-05)

Wednesday 28 January 2026, 11.45h–13.45h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

DON'T PANIC

When a problem seems overwhelming, pause, breathe, and tackle it step by step

---

Problem 1 (5 + 5 + 5 = 15 points)

(a) Define what it means for two sets A and B to have the same cardinality.

(b) Prove that the interval $(0, 1)$ has the same cardinality as the interval $(0, \infty)$.

(c) Is it possible for a sequence $(a_n) \subseteq (0, 1)$ to reach every point in the interval, that is, for all $x \in (0, 1)$ there exists $N_x \in \mathbb{N}$ such that $a_{N_x} = x$? Justify your answer.

Problem 2 (4 + 6 + 5 = 15 points)

Let $x_1 = 7$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right).$$

Prove the following statememts:

(a) $x_n^2 \geq 7$ for all $n \in \mathbb{N}$.

(b) $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. *Hint: what is $x_n - x_{n+1}$?*

(c) x_n converges and $\lim x_n = \sqrt{7}$.

Problem 3 (8 + 7 = 15 points)

(a) Prove that if $A \subseteq \mathbb{R}$ is compact, then for each $\epsilon > 0$ there exists finitely many points $a_1, \ldots, a_N \in A$ such that

$$A \subset V_\epsilon(a_1) \cup V_\epsilon(a_2) \cup \ldots \cup V_\epsilon(a_N).$$

(b) Show that $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ is not compact but satisfies the property nevertheless.

Please turn over for problems 4, 5 and 6!

Problem 4 (9 + 6 = 15 points)

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that f' is bounded, that is, $\exists\, M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous.

(a) Show that the function $h(x) = f(g(x))$ is uniformly continuous on $\mathbb{R}$.

(b) Assume now that f' is *not* bounded. Is h uniformly continuous? If so, prove the statement, otherwise give a counterexample.

Problem 5 (4 + 4 + 7 = 15 points)

Consider the function

$$h(x) = \sum_{n=1}^{\infty} \frac{2}{2\cos^2(nx) + n^2}.$$

Prove the following statements:

(a) The series converges uniformly on $\mathbb{R}$.

(b) h is continuous on $\mathbb{R}$.

(c) h is differentiable on $\mathbb{R}$.

Problem 6 (9 + 6 = 15 points)

Consider the modified Dirichlet function $h : [0, 1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Prove the following statements:

(a) Show that $U(h, P) > \frac{1}{2}$ for any partition P of $[0, 1]$.
Hint: prove that $x_k(x_k - x_{k-1}) > \frac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1})$.

(b) Is h integrable on $[0, 1]$? Justify your answer.

Please do not forget to fill out the online course evaluation!

End of test (90 points)

Note that all the problems could be solved in multiple ways, and not all of those solutions are included here.

Solution of problem 1 (5 + 5 + 5 = 15 points)

(a) A has the same cardinality as B if there exists a bijective function $f : A \to B$, that is, f is one-to-one and onto.

(b) Consider the function $f : (0,1) \to (0,\infty)$ defined by

$$f(x) = \frac{x}{1-x}.$$

This function is bijective with inverse function $f^{-1} : (0,\infty) \to (0,1)$ defined by

$$f^{-1}(y) = \frac{y}{1+y}.$$

This can be shown by explicitly solving $y = f(x)$ and observing that in the domain $1 + y > y > 0$. Therefore $(0,1)$ has the same cardinality as $(0,\infty)$.

(c) No.

Justification 1. Since the interval $(0,1)$ is uncountable, while the sequence (a_n) is countable (as it is indexed by the natural numbers), there exist infinitely many points in $(0,1)$ that are not reached by the sequence (a_n).

Justification 2. Otherwise, (a_n) would provide a bijection between $\mathbb{N}$ and $(0,1)$, which is impossible since $(0,1)$ is uncountable and $\mathbb{N}$ is countable.

Solution of problem 2 (4 + 6 + 5 = 15 points)

(a) Since $x_1^2 = 49 > 7$ the statement is true for $n = 1$.

Now assume that $x_n^2 > 7$ for some $n \in \mathbb{N}$, then

$$\begin{aligned} x_{n+1}^2 - 7 &= \frac{1}{4}\left(x_n^2 + 14 + \frac{49}{x_n^2}\right) - 7 \\ &= \frac{1}{4}\left(x_n^2 + 14 + \frac{49}{x_n^2} - 28\right) \\ &= \frac{1}{4}\left(x_n^2 - 14 + \frac{49}{x_n^2}\right) \\ &= \frac{1}{4}\left(x_n - \frac{7}{x_n}\right)^2 \geq 0, \end{aligned}$$

with equality if and only if $x_n = \sqrt{7}$.

Since $x_n > \sqrt{7}$ by induction hypothesis, it follows that $x_{n+1}^2 - 7 > 0$.

Thus, by induction, $x_n^2 > 7$ for all $n \in \mathbb{N}$.

(b) By part (a) it follows that

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{7}{x_n}\right) \\ &= \frac{x_n}{2} - \frac{7}{2x_n} \\ &= \frac{x_n^2 - 7}{2x_n}. \end{aligned}$$

To show that $x_{n+1} < x_n$ it suffices to show that $x_n > 0$ since we already know from part (a) that $x_n^2 - 7 > 0$.

Either $x_{n+1} > \sqrt{7}$ or $x_{n+1} < -\sqrt{7}$. From the definition of the sequence (x_n), it follows that $x_n > 0$ implies that $x_{n+1} > 0$. Since $x_1 > 0$, it follows that $x_n > 0$ for all $n \in \mathbb{N}$, and this rules out the possibility of having $x_n < -\sqrt{7}$ and thus $x_{n+1} M x_n$ for all $n \in \mathbb{N}$.

(c) By parts (a) and (b) it follows that (x_n) decreases and is bounded from below. The Monotone Convergence Theorem implies that $x = \lim x_n$ exists.

Note that $x = \lim x_{n+1}$ as well. The Algebraic Limit Theorem shows that x satisfies the equation $x = \frac{1}{2}\left(x + \frac{7}{x}\right)$, or, equivalently, $x^2 = 7$. Hence, $x = \sqrt{7}$.

Problem 3 (8 + 7 = 15 points)

(a) Let $\epsilon > 0$ be arbitrary. For any $a \in A$, the set $V_\epsilon(a)$ is an open set.

The collection of open sets $\{V_\epsilon(a) \mid a \in A\}$ is an open cover of A, since $A \subset \cup_{a \in A} V_\epsilon(a)$.

Since A is compact, any open cover has a finite subcover. That is, there exist finitely many points $a_1, \ldots, a_N \in A$ such that

$$A \subset V_\epsilon(a_1) \cup V_\epsilon(a_2) \cup \ldots \cup V_\epsilon(a_N).$$

(b) The set A is not compact since it is not closed: the limit point 0 of A does not belong to A.

Let $\epsilon > 0$ be arbitrary and take $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then $0 \in V_\epsilon(1/N)$ and A has only finitely many elements outside $V_\epsilon(1/N)$ since if $1/n \notin V_\epsilon(1/N)$, then

$$\frac{1}{n} > \frac{1}{N} + \epsilon \quad \implies \quad n < \frac{N}{1 + N\epsilon}.$$

This shows that the noncompact set A can be covered by finitely many of the ϵ-neighbourhoods $V_\epsilon(1/n)$.

Solution of problem 4 (9 + 6 = 15 points)

(a) Since f' is bounded, by the Mean Value Theorem it follows that for all $x, y \in \mathbb{R}$ there exists $c \in (x, y)$ such that

$$|f(x) - f(y)| = |f'(c)(x - y)| \leq M|x - y|.$$

Let $\epsilon > 0$ be arbitrary. Since g is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \implies \quad |g(x) - g(y)| < \frac{\epsilon}{M} \quad \text{for all} \quad x, y \in \mathbb{R}.$$

Therefore, for all $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ we have

$$|h(x) - h(y)| = |f(g(x)) - f(g(y))| \leq M|g(x) - g(y)| < M\frac{\epsilon}{M} = \epsilon,$$

which shows that h is uniformly continuous on $\mathbb{R}$.

(b) No.

For a counterexample one can take $f(x) = x^2$ and $g(x) = x$. Then $f'(x) = x$ is not bounded on $\mathbb{R}$ and $h(x) = f(g(x)) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Solution of problem 5 (4 + 4 + 7 = 15 points)

(a) For all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ we have

$$0 \leq f_n(x) \leq \frac{2}{n^2}, \quad \text{where} \quad f_n(x) = \frac{2}{2\cos^2(nx) + n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, by the Weierstrass M-test with $M_n = \frac{2}{n^2}$ it follows that $h(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.

(b) Each function f_n is continuous on $\mathbb{R}$: this follows from the fact that the denominator of $f_n(x)$ is never zero, the Algebraic Continuity Theorem and the fact that the cosine is a continuous function.

Since the series defining $h(x)$ converges uniformly on $\mathbb{R}$, it follows that it is continuous on $\mathbb{R}$.

(c) Let us compute the derivative of $f_n(x)$:

$$f_n'(x) = \frac{d}{dx}\left(\frac{2}{2\cos^2(nx) + n^2}\right) = \frac{8n\cos(nx)\sin(nx)}{(2\cos^2(nx) + n^2)^2}.$$

This is well defined for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, which in particular shows that each f_n is differentiable on $\mathbb{R}$.

For any $c > 0$,

$$|f_n'(x)| \leq \frac{8n}{(2\cos^2(nx) + n^2)^2} \leq \frac{8n}{n^4} = \frac{8}{n^3} \quad \text{for all } x \in [-c, c].$$

The series $\sum_{n=1}^{\infty} \frac{8}{n^3}$ converges, so by the Weierstrass M-test with $M_n = \frac{8}{n^3}$ it follows that the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[-c, c]$.

Observe that $\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{2}{2+n^2}$ converges.

Therefore, by the theorem on the term-by-term differentiation of series of functions, it follows that h is differentiable on $[-c, c]$. Since c is arbittray, h is differentiable on all $\mathbb{R}$.

Solution of problem 6 (9 + 6 = 15 points)

(a) Let $P = \{0 = x_0 < x_1 < \ldots < x_N = 1\}$ be an arbitrary partition of $[0, 1]$. Observe that

$$M_k = \sup\{h(x) \mid x \in [x_{k-1}, x_k]\} = x_k.$$

Further justification here, using the density of the rationals, is optional.

Therefore, the upper sum of h with respect to the partition P is given by

$$U(h, P) = \sum_{k=1}^{N} M_k(x_k - x_{k-1}) = \sum_{k=1}^{N} x_k(x_k - x_{k-1}).$$

Note that for all $k = 1, \ldots, n$ we have

$$x_k > x_{k-1} \quad \Longrightarrow \quad x_k + x_k > x_k + x_{k-1} \quad \Longrightarrow \quad x_k > \frac{1}{2}(x_k + x_{k-1}).$$

Therefore, the upper sum can be bounded below as follows:

$$\begin{aligned} U(h, P) &= \sum_{k=1}^{N} x_k(x_k - x_{k-1}) > \sum_{k=1}^{N} \frac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1}) \\ &= \frac{1}{2}\sum_{k=1}^{N}(x_k^2 - x_{k-1}^2) = \frac{1}{2}(x_N^2 - x_0^2) = \frac{1}{2}. \end{aligned}$$

(b) No.

Let $P = \{0 = x_0 < x_1 < \ldots < x_N = 1\}$ be an arbitrary partition of $[0, 1]$. Observe that

$$m_k = \inf\{h(x) \mid x \in [x_{k-1}, x_k]\} = 0.$$

Thus, the lower sum of h with respect to the partition P is $L(h, P) = 0$.

Combining this with part (a) it follows that for any partition P of $[0, 1]$ we have

$$U(h, P) - L(h, P) > \frac{1}{2}$$

and thus h is not integrable on $[0, 1]$.